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Generic behavior in linear systems with multiplicative noise

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A classification of the probability distribution of linear multiplicative noise equations is developed which has a variety of physical applications; for example, to the temperature field of a turbulent fluid undergoing a chemical reaction. Without a reaction the tails of the distribution are shown generically to have exponential or stretched exponential behavior. With a reaction occurring, the tails cross over to power-law behavior. A simple criterion in terms of the generalized Lyapunov exponents for the system explain this kind of behavior. Rigorous results for the existence of power-law tails under general symmetry conditions are presented and are easily understood in terms of generalized Lyapunov exponents.

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I. INTRODUCTION

There are many problems in which general classification schemes have proved useful in understanding physical phenomena. Examples of this occur in the study of phase transitions, field theory, and nonlinear dynamics. Nonequilibrium statistical mechanics appears to have less generality than the above cases, although recently there has been progress made in classifying generic behavior of particular types of nonequilibrium systems [1–3].

In this Rapid Communication another category of nonequilibrium system is examined, the case of linear equations containing multiplicative and additive noise. Equations of this type are quite common in physics. For example, the problem of a passive scalar such as a dye, convected by a random velocity field, has been the focus of much recent attention [4–14]:

$$\partial_t \phi + \partial_\mu v_\mu \phi = d \partial_{\mu\mu} \phi. \qquad (1)$$

The (implicit) sums over the index μ are from 1 to the dimension of the system. In two and higher dimensions this is normally studied with the incompressibility condition $\partial_\mu v_\mu = 0$. d is a diffusion constant, and $v = v(\mathbf{r}, t)$ is a random function of position and time. This equation involves multiplicative noise because the random velocity v multiplies the dependent variable ϕ. The Schrödinger equation with a random time-dependent potential is another example [15, 16], and this is closely related to wave propagation in random media [17, 18]. Population growth models which are relevant to chemical reactions [19] and population biology [20, 21] are often of this form

$$\partial_t \phi = \alpha \phi + f \phi + d \partial_{\mu\mu} \phi, \qquad (2)$$

where f is a random function of position and time, and α and d are parameters. Polymers in turbulent flow are modeled by such equations [22, 23].

Additive noise should often be included in these equations. In the case of passive scalar fields, this corresponds to including thermal fluctuations in the velocity of the flow superimposed on the turbulent motion and the effect of boundary conditions on the fluid far from a boundary. For polymers in turbulent flow this corresponds to including the Brownian motion of the individual atoms of the polymer chain. It will be shown analytically that under a certain general symmetry condition the probability distribution for the order parameter ϕ has a power-law tail, $P(\phi) \propto 1/\phi^p$. The exponent characterizing the asymptotic decay of the probability distribution is in general a function of the strength of the multiplicative noise, but not the additive noise. Also this behavior is quite robust in the sense that it is also seen even if the multiplicative noise is not white and is not Gaussian, as will be seen later in this paper. An important question is what determines the existence of power laws for such equations.

When this symmetry condition is violated as it is for passive scalar fields, Eq. (1), then another behavior for the tail of the probability distribution is possible, such as stretched exponential behavior. This stretched exponential behavior is relevant to a variety of experimental situations and has been the focus of intense experimental [4, 5] and theoretical [5, 8, 10–14] investigations. The theoretical approaches taken so far have constructed physical models to explain this intriguing result. The approach taken here is different and argues that this stretched exponential behavior is simply a result of a conservation law for Eq. (1) and the general properties of multiplicative random equations. There is no reference to any detailed physical model. A prediction of this work is that the tail should become a power law when an exothermic chemical reaction takes place during advection of the field.

A related one-component problem has recently been considered by Drummond [24]. The additive and multiplicative noises were taken to be generated by the same random process. In this case two types of behavior are possible. In one regime the probability distribution has power-law tails. The other regime has all its moments defined. He conjectured that this behavior should hold for n components. The case considered in this paper takes the additive and multiplicative noises as independent. This case is amenable to a complete understanding for an n-component system as described below. For the one-component case, the case of arbitrary temporal correlations, or short-range non-Gaussian multiplicative noise, can be fully analyzed. In all these cases it will be seen that the generalized Lyapunov exponents for the system in the absence of additive noise completely determine the behavior with additive noise present. A simple criterion for when power-law tails are present will be derived.

The stability of the nth moment of ϕ in the absence of additive noise has been studied by the introduction of Lyapunov functions [25]. If such a function can be found this provides a method for determining the stability of moments. The addition of additive noise can be incorporated into this method, but does not provide a classification of the kinds of behavior expected for the tails of the distribution.

We will start by considering the coupled differential equations

$$\dot{\boldsymbol{\phi}} = \mathbf{M}\cdot\boldsymbol{\phi} + \mathbf{A}\cdot\boldsymbol{\phi} + \boldsymbol{\eta}. \tag{3}$$

$\boldsymbol{\phi} \equiv (\phi_1, \phi_2, \ldots, \phi_N)$ represents the variables of interest, such as dye density as a function of position. The matrix $\mathbf{M}$ is taken to be random and short range correlated in time $\langle M_{ij}(t)M_{kl}(t')\rangle = 2\Gamma_{ijkl}\delta(t-t')$. $\mathbf{A}$ is taken to be time independent. For the moment we shall consider the case where the additive noise term $\boldsymbol{\eta}$ is zero. The formal solution of this equation is

$$\boldsymbol{\phi}(t) = T\exp\left(\int_0^t \mathbf{M}(\tau) + \mathbf{A}d\tau\right)\boldsymbol{\phi}(0), \tag{4}$$

where T denotes the time ordered product, and is essentially the same as the problem of the multiplication of $t/\Delta t$ random matrices $\exp[\Delta t(\mathbf{M}+\mathbf{A})]$. This is a well studied problem where general theorems have been established. One theorem [28] states that there is a well defined rate of exponential divergence of $|\boldsymbol{\phi}(t)|$ for sufficiently long times. Higher moments of ϕ such as ϕ_i^2 also show exponential divergence but, in general, with different exponents. One can define a generalized Lyapunov exponent $L(q)$ by [26, 27]

$$\langle \phi_i^q\rangle \propto e^{L(q)t}, \tag{5}$$

where the brackets denote an ensemble average over the noise $\mathbf{M}$. From this it can be seen that the probability distribution for a component of $\boldsymbol{\phi}$, say, ϕ has a probability distribution

$$\ln P(\ln\phi) \propto t f(\ln\phi/t) + O\left(\frac{\ln t}{t}\right). \tag{6}$$

This can also be seen by making thermodynamic analogies [26, 27], and is quite similar to the discussion of $f(\alpha)$ in the literature on multifractals.

For a system described by such a multiplicative noise equation, such as dye diffusion in a random velocity field, such behavior should be observed for long times, for the probability distribution for dye density.

One can obtain the different $L(q)$'s as being the the lowest eigenvalues for a set of linear operators as follows. Using standard methods [29] it is possible to derive the equations for the moments of ϕ. For the specific case of the population growth model, this has been recently done in Ref. [21]. Define $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\boldsymbol{\beta} \equiv (\beta_1, \beta_2, \ldots, \beta_n)$ where the α_i's and β_i's can take on any integral value between 1 and N. Define $\phi_{\boldsymbol{\alpha}} \equiv \langle \phi_{\alpha_1}\phi_{\alpha_2}\cdots\phi_{\alpha_n}\rangle$. Then the equation for the nth moment can be written in the form

$$\sum_{\boldsymbol{\beta}}(a_{\boldsymbol{\alpha},\boldsymbol{\beta}} - G_{\boldsymbol{\alpha},\boldsymbol{\beta}})\phi_{\boldsymbol{\beta}} = -\frac{d\phi_{\boldsymbol{\alpha}}}{dt}, \tag{7}$$

where

$$a_{\boldsymbol{\alpha},\boldsymbol{\beta}} = -\sum_{i=1}^{n}\left(\prod_{\substack{j\\ j\neq i}}^{n}\delta_{\alpha_j,\beta_j}\right)A_{\alpha_i,\beta_i}, \tag{8}$$

and

$$G_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{i=1}^{n}\sum_{\gamma=1}^{N}\Gamma_{\alpha_i\gamma\gamma\beta_i}\prod_{\substack{j\\ j\neq i}}^{n}\delta_{\alpha_j,\beta_j} + \sum_{\substack{i,j\\ i<j}}[\Gamma_{\alpha_i\beta_i\alpha_j\beta_j} + \Gamma_{\alpha_i\beta_j\alpha_j\beta_i}]\prod_{k\neq i,j}^{n}\delta_{\alpha_k,\beta_k}. \tag{9}$$

The long time rate of divergence of $\phi_{\boldsymbol{\alpha}}$, $L(q)$, is given by the lowest eigenvalue in Eq. (7). Note that generically all components of $\phi_{\boldsymbol{\alpha}}$ diverge with the same exponent.

Next we examine the behavior of systems of the type described by Eq. (3) when there is white noise η included in Eq. (3). η is taken to be white and uncorrelated $\langle \eta_i(t)\eta_j(t')\rangle = D\delta(t-t')\delta_{i,j}$.

We first motivate the discussion by examining the special case of one component, $N=1$, where the probability distribution can be solved exactly by standard means by writing down the Fokker-Planck equation describing its time evolution [29]. In the steady state the problem reduces to solving a second-order linear ordinary differential equation with the appropriate boundary conditions. One obtains

$$P(\phi) \propto \frac{1}{(\Gamma\phi^2 + D)^{\frac{1}{2}-A/\Gamma}}, \tag{10}$$

which shows a power-law tail that depends continuously on A and Γ and is independent of the strength of the additive noise D. The result here is similar to the three-dimensional case examined in the context of polymers in turbulent flow [22].

Returning to the more general case of N variables, the moments in steady state have values

$$\phi_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}}(\mathbf{a}-\mathbf{G})^{-1}{}_{\boldsymbol{\alpha},\boldsymbol{\beta}}N_{\boldsymbol{\beta}}, \tag{11}$$

where $\mathbf{N}$ is

$$N_{\boldsymbol{\alpha}} = D\sum_{\substack{i,j\\ i<j}}\delta_{\alpha_i,\alpha_j}\phi_{\alpha_1,\ldots\alpha_{i-1},\alpha_{i+1},\ldots\alpha_{j-1},\alpha_{j+1},\ldots\alpha_N}. \tag{12}$$

Having a power-law tail in the probability of a component of $\boldsymbol{\phi}$ is equivalent to having $\phi_{\boldsymbol{\alpha}}$ diverge at some finite n. This corresponds to the point where the lowest eigenvalue of $\mathbf{a}-\mathbf{G}$ passes through zero. Since the lowest eigenvalue of $\mathbf{a}-\mathbf{G}$ equals $-L(q)$, then the first value of q, q^*, which has divergent moments is where $L(q^*)=0$. The behavior of the probability distribution for large ϕ is therefore $P(\phi) \propto \phi^{-q^*-1}$.

It is straightforward to demonstrate that q^* is finite [30] when we restrict $\mathbf{M}(t)$ to be invertible and either real symmetric or real antisymmetric. If one makes the choice $\hat{\boldsymbol{\phi}} \propto (1,1,\ldots,1)$, then the scalar product $\hat{\boldsymbol{\phi}}\cdot(\mathbf{a}-\mathbf{G})\cdot\hat{\boldsymbol{\phi}}$ can be evaluated. For sufficiently large n it goes negative, showing that at least one eigenvalue of $\mathbf{a}-\mathbf{G}$, and hence $L(q)$, has gone through zero, giving power-law tails.

The above argument shows that for a system described by Eq. (3), and with the correct symmetry, there must always be moments of the ϕ's that diverge if sufficiently high moments are examined. All of the moments of a given order are expected to diverge at the same time, if the matrix Γ is not of block diagonal form. This means that $\langle\phi_1^n\rangle$ should diverge at the same n as $\langle\phi_2^{n-2}\phi_3^2\rangle$, for example. Note that the divergence is not dependent on D, the strength of the additive noise, and thus the power law should also be unaffected by this. The power-law tail should only depend on $\boldsymbol{\Gamma}$ and $\mathbf{A}$.

If the random matrix $\mathbf{M}$ is not either symmetric or antisymmetric then there is no guarantee that $L(q)$ will pass through zero. In fact, for Eq. (1), $L(q)$ should remain negative for all q. This can be seen easily as follows. First, $\int\phi(r)d^dr$ taken over the entire volume is conserved, and second, if $\phi(r)$ is positive for all r it must remain so. Therefore for a discretized lattice model, there is a maximum to the value that $\phi(r)$ can achieve and therefore $L(q)$ can never become positive. Off lattice the viscosity acts as a short distance cutoff and should give the same result.

If a term $\alpha\phi$ is added to the right hand side of Eq. (1) then this has the effect of shifting $L(q)$ to $L(q)+\alpha q$. In this case, for any $\alpha>0$, $L(q)$ will intersect the q axis so that power-law tails for the probability distribution will be seen. We shall return to this case below.

The above arguments assume that the randomness inherent in the matrix $\mathbf{M}$ is Gaussian and short range. It is possible to analyze both assumptions separately for a one-component system and show that the same power-law behavior should be present when these assumptions are relaxed. Consider the equation

$$\dot{\phi} \;=\; f(t)\phi+\eta(t). \tag{13}$$

One can consider the case where (a) $f(t)$ is Gaussian and has arbitrary correlations in time, and (b) $f(t)$ is short range in time, but is not Gaussian. For case (a) the power-law tail can be computed [30] and is $P(\phi)\propto\phi^{-p}$, where

$$p \;=\; 1/2-\langle f(t)\rangle\Big/\int_{-\infty}^{\infty}\langle f(t)f(0)\rangle_C dt. \tag{14}$$

Therefore the power-law tail only depends on the power spectrum at zero frequency.

Case (b) has also been analyzed. In the absence of η, that is, additive noise, one can characterize the randomness f by having it give rise to generalized Lyapunov exponents $L(q)$ for the qth moments, as in Eq. (5). Then in the presence of additive noise, the steady-state value of the $2n$th moment is [30]

$$\langle\phi^{2n}\rangle \;=\; (2n!)\left(\frac{D}{2}\right)^n\prod_{i=1}^{n}\frac{1}{-L(2i)}. \tag{15}$$

For $2n>q*$ this expression diverges. This shows that the critical n at which moments diverge is identical to that found above in the multicomponent case with Gaussian noise.

Is there an underlying reason why the multidimensional Gaussian case should give the same result as the one-dimensional non-Gaussian one? A simple heuristic argument can be given to support the notion that the multidimensional case is essentially one dimensional in character. If one considers the case of no additive noise, then the vector $\boldsymbol{\phi}$ will randomly vary as a function of time. One can normalize $\boldsymbol{\phi}$ defining $\boldsymbol{\phi}' = \boldsymbol{\phi}/|\boldsymbol{\phi}|$. Then $\boldsymbol{\phi}'$ will settle down at some time, $t=t_0$, to some "typical" conformation. For example, in the case of passive scalar fields, the dye density will have some well defined equilibrium correlations. After a time, $t=t_0+T_1$, where T_1 is perhaps many relaxation times, $\boldsymbol{\phi}'$ will become close to what it was originally. At this point the situation is similar to what is at $t=t_0$ as far as $\boldsymbol{\phi}'$ is concerned, but now $|\boldsymbol{\phi}|$ has changed. Because the equation of motion is linear, it will evolve at time $t=t_0+T_1$ in the same way as it did at time t_0, except now with a new realization of the random noise $\mathbf{M(t)}$. At some still later time $t=t_0+T_2$, $\boldsymbol{\phi}'$ is once again close to its original configuration, but now $|\boldsymbol{\phi}|$ is the product of two independent random variables, $|\boldsymbol{\phi}(t_0+T_2)|/|\boldsymbol{\phi}(t_0+T_1)|$ and $|\boldsymbol{\phi}(t_0+T_1)|/|\boldsymbol{\phi}(t_0)|$. Therefore the evolution of $|\boldsymbol{\phi}|$ can be obtained by the multiplication of independent random numbers that are distributed in a highly non-Gaussian way, which depends on the detailed dynamics of the system within a relaxation time.

Last we examine what happens at the crossover between power-law behavior and non-power-law behavior for $P(\phi)$. As mentioned above, if $L(q)$ is always negative for $q>0$, one can add a term $\alpha\phi$ to the equation of motion. For sufficiently large α, $L(q)$ must cross the origin, because $L(q)$ is convex. What happens at the the critical α where this crossover is about to occur? One can classify what happens according to the behavior of $L(q)$ at the transition point. For the critical value of α it has not crossed the origin, and being convex, the large q behavior must be slower than linear. For large q one can consider two classes. (i) Consider the class $L(q)\propto -q^\beta$ with $0\le\beta\le1$. Substituting this into Eq. (15) and solving for $P(\phi)$, for large ϕ one obtains

$$P(\phi)\propto e^{-K\phi^{2/(2-\beta)}}, \tag{16}$$

where K is a constant. (ii) When $L(q)$ approaches a constant for large q, $P(\phi)\propto e^{-K\phi}$. Note that for class (i) $f(\alpha)$ [cf. Eq. (6)] goes to $-\infty$, and for class (ii) $f(\alpha)$ stops at a finite value.

As mentioned above, Eq. (1) should not give power-law tails when additive noise is included. If a model such as the β model for turbulence applies to this situation [26], then the behavior is pure exponential, as in case (ii) above. However, when a term $\alpha\phi$ is added to the right hand side, power-law tails are expected. Figure 1 shows the probability distribution of the one-dimensional version of Eq. (1) on a log-log plot, obtained by simulation. The incompressibility constraint is not enforced in one dimension. The value of the power law seen depends on α. A system with nonzero α could be realized experimentally, by having an exothermic reaction [19] occurring in a fluid undergoing random or turbulent motion. If molecules change irreversibly from type A to B, the rate constant will in general depend on temperature. From this it can be seen that the rate of change in temperature has a term proportional to temperature, for small deviations in the temperature. The temperature evolution should therefore be well described by Eq. (1) with the addition of an $\alpha\phi$ term and additive noise to the right hand side. The power-law exponent should be a function of the system size and of the type and strength of random stirring. If the system is forced by heating of a boundary, then the heat shed from it should act as an additive noise source in addition to the thermal noise already mentioned. The strength of such a noise is expected to be much greater than thermal noise. It is important in this situation to keep the size of the system below the critical size where an exponential increase in the mean temperature occurs. Small Rayleigh-Bénard cells with a reactive fluid in the soft turbulent regime probably offer the easiest test of this power-law prediction.

In conclusion, linear equations with multiplicative noise and additive noise often lead to power-law tails in the steady-state probability distribution of a field; for example, when the random matrix characterizing the multiplicative noise is Gaussian and either symmetric or antisymmetric. This is true for systems with an arbitrary number of components. The power law should depend continuously on all the parameters but the additive noise. When this symmetry condition is not satisfied, or the noise is not Gaussian or short range, power-law tails are still expected in a large number of circumstances. The general criterion is that the generalized Lyapunov exponent $L(q)$ should become positive for large enough q. If this is not the case, as in Eq. (1), then the behavior should be either exponential or stretched exponential. By adding a linear term $\alpha\phi$ to such an equation, which for example could correspond physically to a reactive fluid, power-law tails should be seen.

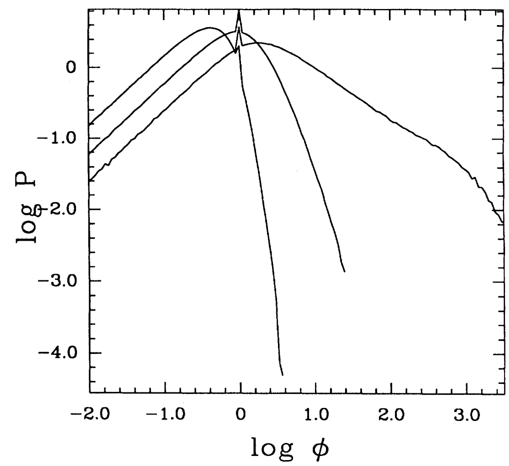

FIG. 1. The probability distribution of field ϕ obeying Eq. (1) in one dimension, when a term $\alpha\phi$ and additive noise are included on the right hand side. Three different values of α are shown. The most steeply descending curve is for $\alpha = 0$, and the next is $\alpha = 0.47$. The top curve is for $\alpha = 0.51$. The equation was discretized and the number of lattice sites was chosen to be 8. The log base is 10.

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